$\frac{29}{30}$

Differential Calculus

Lab Assignment 7

Show ALL your work to receive FULL credit.

1. (5 marks) Find $f'(0)$, where $f(x) = \begin{cases} x^9 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

2. Use the definition to find the derivatives of the following functions:

   (a) (6 marks) $f(x) = \dfrac{1}{x^2}$

   (b) (6 marks) $f(x) = \sqrt{x+1}$

   (c) (6 marks) $f(x) = \sec x$

3. Differentiate the function.

   (a) (3 marks) $f(x) = e^x \tan x$

   (b) (4 marks) $f(x) = \dfrac{1 - xe^x}{x + e^x}$

1) $f(x) = \begin{cases} x^9 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

$$f'(0) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

$$= \lim_{h \to 0} \frac{h^9 \sin(1/h) - 0}{h}$$

5

$$= \lim_{h \to 0} h^8 \sin(1/h)$$

$$-1 \le \sin\frac{1}{h} \le 1$$

$$-h^8 \le h^8 \sin\frac{1}{h} \le h^8$$

$$\lim_{h \to 0} -h^8 \le \lim_{h \to 0} h^8 \sin\frac{1}{h} \le \lim_{h \to 0} h^8$$

$$-0^8 \le \lim_{h \to 0} h^8 \sin\left(\frac{1}{h}\right) \le 0^8$$

$$0 \le \lim_{h \to 0} h^8 \sin\left(\frac{1}{h}\right) \le 0$$

$\therefore$ derivative is zero at 0

2) a) $\displaystyle \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h \to 0} \frac{\left(\frac{1}{(x+h)^2}\right) - \left(\frac{1}{x^2}\right)}{h}$

5

$$\lim_{h \to 0} \frac{\frac{x^2}{(x+h)^2} - \frac{(x+h)^2}{x^2}}{h} = \lim_{h \to 0} \frac{x^2 - (x+h)^2}{h(x+h)^2(x^2)}$$

$$= \lim_{h \to 0} \frac{x^2 - (x^2 + 2xh + h^2)}{h(x^2+2xh+h^2)(x^2)} = \lim_{h \to 0} \frac{\cancel{x^2} - \cancel{x^2} - 2xh - h^2}{h(x^2+2xh+h^2)(x^2)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-2x-h)}{\cancel{h}(x^2+2xh+h^2)(x^2)} = \frac{-2x-\cancel{0}}{x^4 + \cancel{2x^3(0)} + \cancel{(0)^2x^2}}$$

$$\boxed{= \frac{-2\cancel{x}}{x^3}}$$

2b) $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h}$

$= \lim_{h \to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \times \frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}}$

6

$= \lim_{h \to 0} \frac{(\sqrt{x+h+1})^2 - (\sqrt{x+1})^2}{h(\sqrt{x+h+1} + \sqrt{x+1})} = \lim_{h \to 0} \frac{x+h+1-(x+1)}{h(\sqrt{x+h+1} + \sqrt{x+1})}$

$= \lim_{h \to 0} \frac{\cancel{x+h+1-x-1}}{\cancel{h}\sqrt{x+h+1} + \sqrt{x+1}} = \frac{1}{\sqrt{x+0+1} + \sqrt{x+1}}$

$= \frac{1}{\sqrt{x+1} + \sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}$

2c) $(\cos x)' \to \lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h} = \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$

$= \lim_{h \to 0} \frac{(\cos x \cos h - 1) - \sin x \sin h}{h} = \lim_{h \to 0} \frac{\cos x \cos h - 1}{h} - \frac{\sin x \sin h}{h}$

6

$= \lim_{h \to 0} \frac{\cos h - 1}{h} \cdot \cos x - \frac{\sin h}{h} \cdot \sin x$

$= -\sin x$

$(\sec x)' = \left(\frac{1}{\cos x}\right)'$

$= \lim_{h \to 0} \frac{(0)(\cos x) - (1)(-\sin x)}{\cos^2 x}$

$= \lim_{h \to 0} \frac{\sin x}{\cos^2 x} = \lim_{h \to 0} \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \tan x \sec x$

3a) $f(x) = \overbrace{e^x}^{f(x)} \overbrace{\tan x}^{g(x)}$

Product rule

$f'(x)g(x) + f(x)g'(x)$

3

$(e^x \tan x)' = (e^x)'(\tan x) + (e^x)(\tan x)'$

$= (e^x)(1)(\tan x) + (e^x)(\sec^2 x)$

$= (e^x)(\tan x) + (e^x)(\sec^2 x)$

$\boxed{= e^x(\tan x + \sec^2 x)}$

3b) $f(x) = \dfrac{1 - xe^x \} \to f(x)}{x + e^x \} \to g(x)}$

$f'(x)g(x) + f(x)g'(x)$

$f'(x) = \dfrac{(1)' - (xe^x)'}{x + e^x} = \dfrac{0 - ((x)'(e^x) + (x)(e^x)')}{x + e^x}$

Quotient Rule $\to \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

4

$= \dfrac{(-xe^x - e^x)(x + e^x) - (1 - xe^x)(x + e^x)'}{(x + e^x)^2}$

$= \dfrac{(-xe^x - e^x)(x + e^x) - (1 - xe^x)(1 + e^x)}{(x + e^x)^2}$