[redacted] Linear Algebra I
Midterm exam 1, [redacted]

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This exam duration is 70 minutes. It consists of 3 problems and 9 pages. The space below each question should be enough for the answer. Also, the number of points allocated to each question should be an indication on the expected time necessary to answer that question. Please make sure to verify your answers.

Problem	Grade out of
1	33 / 40
2	5 / 35
3	11 / 25
Total	49 /100

1. (40 points)

(a) (10 pts) Use Gauss Elimination to solve the following system of equations:

$$\begin{array}{rrrl} x_1 & & -x_3 & = -2 \\ -2x_1 & +x_2 & +x_3 & = 3 \\ -x_1 & +x_2 & -x_3 & = -2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ -2 & 1 & 1 & 3 \\ -1 & 1 & -1 & -2 \end{array}\right]$$

① $-R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ -2 & 1 & 1 & 3 \\ 1 & -1 & 1 & 2 \end{array}\right]$ ② $\frac{1}{2}R_2$, R_2+R_1, $R_3-R_1 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1/2 & -1/2 & 1/2 \\ 0 & -1 & 2 & 2 \end{array}\right]$

③ $\frac{1}{2}R_3$, $2R_2$, $R_3 + 1/2 R_2$, $2R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$\boxed{x_3 = 3}$

$x_2 - x_3 = 1$

$x_2 - 3 = 1$

$x_2 = 1+3$

$\boxed{x_2 = 4}$

-3

$x_1 - x_3 = -2$

$x_1 - 3 = -2$

$x_1 = -2+3$

$\boxed{x_1 = 1}$

(b) (5 pts) Use Gauss-Jordan Elimination to solve the system of equations in part (a).

Starting from the Gauss form above → $\left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$R_2+R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$R_1+R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$\boxed{\begin{array}{l} x_3 = 3 \\ x_2 = 4 \\ x_1 = 1 \end{array}}$

-1

(c) (13 pts) Use Gauss-Jordan elimination to find A^{-1}, if it exists, where $A = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{then } -\frac{1}{2}R_3]{R_2+R_1, \text{ then } R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right]$$

$$\begin{matrix} R_2+R_1 \\ R_3+1/2R_2 \\ 2R_3 \end{matrix} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right] \xrightarrow[R_1+R_3]{R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & 3 & 2 & -1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

$\therefore$, the inverse is $\begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & -1 \\ 1 & 1 & -1 \end{bmatrix}$

(d) (7 pts) Determine whether the following vectors are linearly independent. Justify your answer.

$$v^1 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}, v^2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \text{ and } v^3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

$$\alpha \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

① $\alpha + \gamma = 0 \rightarrow \gamma = -\alpha$

② $-2\alpha + \beta - \gamma = 0 \rightarrow \gamma = -2\alpha + \beta$

③ $-\alpha + \beta = 0$

α:

$-\alpha = -2\alpha + \beta$

$\alpha = \beta$

Sub in ③ $-\alpha + \beta = 0$

$-(\beta) + \beta = 0$

$0 = 0$

The only solution is $0=0$ for α, β, γ. $\therefore$, it is linearly independant

γ:

$\gamma = -\alpha$

$-2(-\gamma) + \beta - \gamma = 0$

$2\gamma + \beta - \gamma = 0$

$\gamma + \beta = 0 \rightarrow \beta = \gamma + 2\alpha$

$\gamma + \gamma + 2(-\gamma) = 0$

$2\gamma + (-2\gamma) = 0$

$0 = 0$

-3

(e) (5 pts) For what value of β are the following vectors linearly dependent. Justify your answer.

$$u^1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, u^2 = \begin{bmatrix} 6 \\ 3 \\ \beta \end{bmatrix}.$$

$$\left[\begin{array}{cc|c} 2 & 6 & 0 \\ 1 & 3 & 0 \\ -1 & \beta & 0 \end{array}\right]$$ (There should be a non zero Solution to β)

β will be linearly dependant if u^1 and u^2 are not Linear combinations

$3u_1 = u_2$

$3\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ \beta \end{bmatrix}$

$\begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ \beta \end{bmatrix}$

$\beta \neq -3$ or they're a linear combination

$\therefore$, β can be anything but -3

2. (35 points) Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$ and its reduced row echelon form $\text{rref}(A) = \begin{bmatrix} \boxed{1} & 0 & 3 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & 0 \end{bmatrix}$.

(a) (8 pts) Find the general (or complete) solution of $Ax = b$ where $b = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & +1 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_3 - R_1$

$-\frac{1}{2}R_3$

$R_3 - 1/2 R_2$

$-R_2$

$$\boxed{\begin{array}{l} X_2 - 2X_3 = -2 \\ X_1 + X_2 + X_3 = 3 \end{array}}$$

1

(b) (3 pts) What is rank A. Is A nonsingular, i.e., does A^{-1} exists?

There is no inverse because in rref you get one row of zeroes. The rank is 2 (2 pivots)

singular

2

(c) (7 pts) Find a nonzero vector x in $N(A)$, the null space of A.

Set free variable = 1

The free variable is X_3

$X_2 - 2(1) = -2$

$X_2 - 2 = -2$

$X_2 = -2 + 2$

$\boxed{X_2 = 0}$

$X_1 + X_2 + 1 = 3$

$X_1 + 0 + 1 = 3$

$X_1 + 1 = 3$

$X_1 = 3 - 1$

$\boxed{X_1 = 2}$

0

The vector is $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

(d) (7 pts) Let $b = \begin{bmatrix} 2 \\ 1 \\ b_3 \end{bmatrix}$. For what values of b_3 is the vector b in $C(A)$, the column space of A?

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & \left(-\frac{1}{2}(b_3-2)\right)-\frac{1}{2} \end{array}\right]$$

$R_3 - R_1$
$-\frac{1}{2}R_3$
$R_3 - \frac{1}{2}R_2$
$-R_2$

$b_3 = 0$, then vector b would be in the $C(A)$, because the last row is zeroes and

$0+0+0 \neq$ a number, so b_3 must $= 0$ ✗

0

(e) (10 pts) Let $a^1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $a^2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$. Find a nonzero vector y in $\mathbb{R}^3$ that is perpendicular to both a^1 and a^2.

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 2 & -1 \end{array}\right]$$

$$\text{rref} = \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = 3$
$x_2 = -2$

$$\begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$$ ✗

2

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow$$ all zeroes so this doesn't count

$Y_1 = 0$
$Y_2 = 0$
$Y_3 = 0$

3. (25 points)

(a) (7 pts) Find matrix A such that

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ 3 & -2 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \times$$

$R_1 + 2R_2$

$$\begin{bmatrix} 1+0 & -2+0 \\ 0 & -2+1 \end{bmatrix} = \begin{bmatrix} \cancel{1} & -2 \\ 0 & -1 \end{bmatrix} A = \begin{bmatrix} -5 & 4 \\ 3 & -2 \end{bmatrix}$$

1

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{array}\right] \rightarrow \begin{bmatrix} -5 & 4 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \times$$

(b) (7 pts) Is the set S of vectors $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that $x_3 = x_1 + x_2$ a subspace of $\mathbb{R}^3$? Justify your answer.

1) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \rightarrow$ contains zero vector ✓ ✓

2) $\alpha \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \times \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \rightarrow X_3 = X_1 + X_2$ ✓

$\alpha = 2$ for ex.

2

3) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \times \begin{bmatrix} 3 \\ 7 \\ 10 \end{bmatrix}$ ✓ closed under addition

$\boxed{\text{✓ yes, ✓ it is}}$

(c) (3 pts) Is the set S_2 of nonsingular (invertible) 2×2 matrices a subspace of the space of 2×2 matrices? Justify your answer.

No → it does not contain the zero vector because if it is invertible, then it cannot contain the zero vector ✓

3

(d) (8 pts) Is the set S_3 of 2×2 matrices $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ such that $a_{11} = a_{12} = a_{21} = a_{22}$ a subsapce of the space of 2×2 matrices? Justify your answer.

1) Zero vector → Yes, it could contain that if $a_{11} = a_{12} = a_{21} = a_{22}$ ✓

2) $\alpha \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ → when multiplied by anything, you will get the same values (same ratio) of a_{11} to a_{12} to a_{21} to a_{22} so they all still equal ✓

5

3) $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} 2a_{11} & 2a_{12} \\ 2a_{21} & 2a_{22} \end{bmatrix}$ X → $\begin{bmatrix} 3a_{11} & 3a_{12} \\ 3a_{21} & 3a_{22} \end{bmatrix}$

= Yes, the same ratio holds and $a_{11} = a_{12} = a_{21} = a_{22}$ ✓

∴, it is a subspace